JET PROPULSION LABORATORY

INTEROFFICE MEMORANDUM
3411-95-200csi
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FROM: J. W. Melody

SUBJECT: Discrete-Frequency and Broadband Reaction Wheel Disturbance Models

SUMMARY:

This memo summarizes disturbance modeling of the Hubble Space Telescope (HST) reaction wheels, generated from testing performed on the actual HST flight units. The disturbances are modeled as sinusoidal components at harmonic frequencies of the reaction wheel speed, resulting in a discrete-frequency model. Unfortunately this model does not lend itself easily to broadband frequency domain analysis and loop-shaping control design. From the original, discrete-frequency model, a stochastic broadband model is generated. Both models have been implemented in Matlab, and are available on the jpl-gnc network.

Introduction

In performing pointing and integrated controls-structures-optics performance analyses, it is necessary to model disturbances. For distributed opto-mechanical spacecraft with very tight performance requirements, disturbance modeling becomes critical: a conservative model may result in an excessively costly and complex design, whereas an optimistic model may yield a design that won't meet the requirements.

Oftentimes the only significant disturbances in the medium to high frequency regime are reaction wheel assembly (RWA) disturbances. For this reason, a disturbance model based on testing of the HST RWAs has been used in the performance analysis of several distributed optics space missions (POINTS, OSI, DLI, and SONATA, among others). As described below, this model is a harmonic disturbance model, *i.e.*, it models the disturbance forces and torques as harmonics of the reaction wheel speed.

Since several of these missions employ optical control systems, it is necessary to design and evaluate the control systems in order to address the performance analysis. Since the harmonic model results in a discrete-frequency spectrum, it does not lend itself easily to broadband frequency domain analysis and loop-shaping control design. In order to simplify the control design and analysis, a broadband stochastic model was created from the harmonic model.

HST RWA Harmonic Disturbance Model

Since the Hubble Space Telescope had very fine pointing and mechanical stability requirements, a detailed and accurate disturbance model was needed. For this reason, disturbance characterization tests were performed on the RWA flight units [1]. Based on this test data, the disturbance forces and torques were modeled as consisting of discrete harmonics of the reaction wheel speed, f_{rwa}, with amplitudes proportional to the square of the wheel speed:

$$m(t) = \sum_{i=1}^{n} C_i f_{rwa}^2 \sin\left(2\pi h_i f_{rwa} t + \phi_i\right) \tag{1}$$

where $m(t)$ is the disturbance torque or force, C_i is an amplitude coefficient, h_i is the harmonic number, and ϕ_i is a random phase (uniform over $[0,\ 2\pi]$). Using this model, estimating the amplitude coefficient and the harmonic number is equivalent to determining the amplitude and frequency of each component as a function of wheel speed.

The disturbances measured were forces in the plane of the wheel (radial forces), force along the wheel's axis of rotation (axial force), and wobble torques (radial torques). Torque about the axis of rotation (torque ripple and motor cogging) was found to be insignificant. The resultant model parameters are listed in Table 1 for each disturbance direction. The 1x harmonics are primarily caused by dynamic and static wheel imbalances, while the other harmonics are caused by bearing related sources (*e.g.,* geometric imperfections). For a more detailed discussion of the sources of disturbances, see Appendix F of [1].

Typical Integrated Performance Analysis

A typical integrated controls-structures-optics performance analysis is described in this section for two reasons: 1) to demonstrate the nature of the discrete-frequency disturbance model and 2) to illustrate the need for a broadband RWA disturbance model. In this particular example, the performance metric is the optical path difference (OPD) between two arms of an optical interferometer.

Since the RWA disturbances are dependent upon the wheel speed, a wheel speed must be given in order to perform a disturbance analysis. Unfortunately, the wheel speed during observations will vary as the spacecraft attitude control system reacts to external torques. The extent to which this variation occurs is dependent upon the characteristics of the momentum management system and the external torques. Furthermore, if the wheels are biased at different speeds, momentum will be transferred between wheels during a slew. This results in a change of *separate* wheel speeds after a slew, even though the overall angular momentum remains unchanged by the slew. For these reasons, the wheel speeds can vary over a wide range. Therefore, the performance analysis must be parameterized by the wheel speed.

The parameterized result of this example disturbance analysis is given Figure 1. Figure 1 displays the root-mean-squared OPD variation, σ_{opd}, as a function of wheel speed. *Each point* on the graph represents a standard deviation of OPD resulting from disturbances of a single wheel spinning at the given wheel speed. Figure 1 is not a power spectral density. Instead, each point represents the standard deviation of a discrete-frequency power spectral density.

For the sake of simplicity, it is desirable to condense the performance displayed parametrically in Figure 1 to a single number. The most obvious simplification is to take the largest value of σ_{opd}, analogous to the $\mathcal{H}_\infty$ norm. This can be easily read off of the graph. Another, perhaps more obscure metric would be the root-mean-square of σ_{opd} over the wheel speeds, analogous to the $\mathcal{H}_2$ norm (actually an $\mathcal{L}_2$ norm). The value of this metric is shown in the y-axis label of Figure 1. At first hearing this metric sounds strange, so a mathematical description is in order:

$$\|\sigma_{opd}\|_{\mathcal{L}_2}^2 = \frac{1}{f_{max}} \int_0^{f_{max}} \sigma_{opd}^2 \left(f_{rwa}\right) df_{rwa} \tag{2}$$

where $\|\sigma_{opd}\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm and f_{max} is the maximum wheel speed. This metric was first applied to the Focus Mission Interferometer by S. Sirlin [2].

Radial Forces		Axial Forces	Radial Torques
h_i	$C_i\left(\frac{\mathrm{N}}{(\mathrm{Hz})^2}\right)$	$C_i\left(\frac{\mathrm{N}}{(\mathrm{Hz})^2}\right)$	$C_i\left(\frac{\mathrm{Nm}}{(\mathrm{Hz})^2}\right)$
0.35	2.58x10^{-5}	1.44x10^{-6}	
1.00	1.50x10^{-4}	6.10x10^{-5}	1.93x10^{-5}
2.00	7.91x10^{-5}	9.05x10^{-5}	1.51x10^{-5}
2.82	1.69x10^{-4}	3.09x10^{-4}	7.60x10^{-5}
3.00	2.45x10^{-5}	0	0
3.12	3.01x10^{-5}	4.25x10^{-5}	0
3.25	5.29x10^{-5}	0	0
3.60	2.34x10^{-5}	0	0
3.84	6.82x10^{-5}	5.33x10^{-5}	0
4.00	4.12x10^{-5}	3.99x10^{-5}	0
4.14	1.04x10^{-4}	7.24x10^{-5}	0
4.35	0	7.21x10^{-5}	0
4.42	0	5.78x10^{-5}	0
4.55	4.85x10^{-5}	0	0
4.74	4.69x10^{-5}	0	0
5.00	3.00x10^{-5}	0	0
5.18	8.55x10^{-5}	3.87x10^{-4}	1.46x10^{-4}
5.60	1.60x10^{-4}	3.16x10^{-4}	0
5.76	8.89x10^{-5}	0	0
6.00	2.23x10^{-4}	0	0
6.17	0	3.14x10^{-4}	0
6.64	0	2.25x10^{-4}	0
7.50	1.03x10^{-4}	1.20x10^{-4}	0
8.28	1.46x10^{-4}	0	0
8.50	1.71x10^{-4}	3.32x10^{-4}	0
8.70	1.89x10^{-4}	0	0
9.00	1.20x10^{-4}	0	0
10.20	1.37x10^{-4}	0	0
10.28	0	5.06x10^{-4}	0
10.44	1.20x10^{-4}	0	0
10.80	1.43x10^{-4}	0	0
11.22	3.57x10^{-4}	0	0
11.29	0	2.07x10^{-4}	0
11.78	0	5.70x10^{-4}	0
11.88	2.86x10^{-4}	0	0
14.62	0	3.80x10^{-4}	0

Table 1: Hubble Space Telescope reaction wheel discrete-frequency disturbance model harmonic numbers, h_i, and coefficients, C_i.

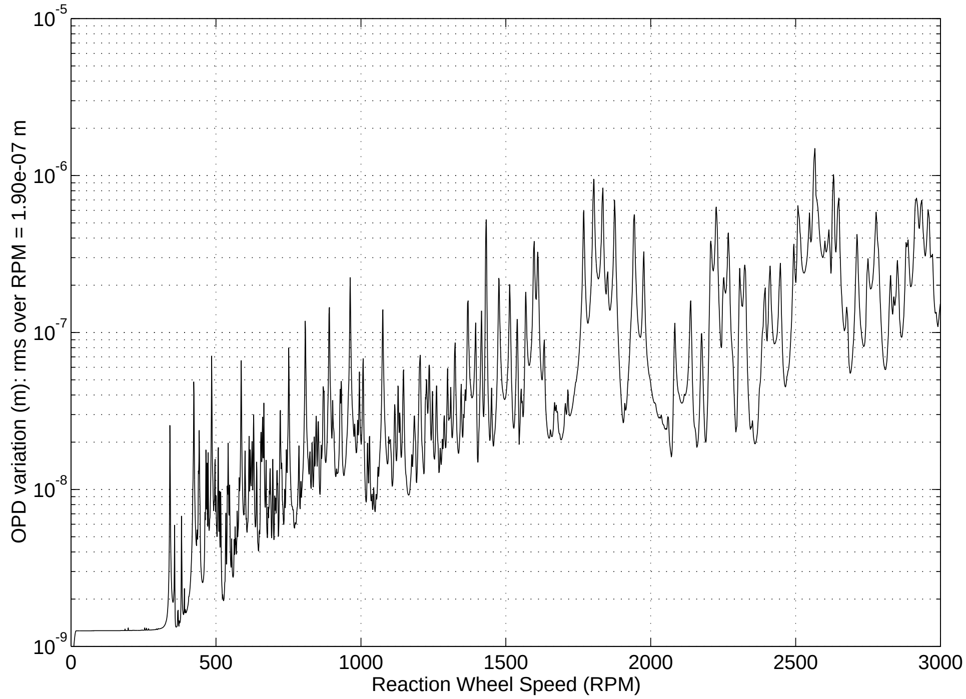

Figure 1: Typical results of a discrete-frequency disturbance analysis showing root-mean-squared OPD as a function of RWA speed.

Why use the $\mathcal{L}_2$ norm at all? It will be shown below that this metric can be calculated without performing the parameterized disturbance analysis, which can be very numerically costly. On the other hand, finding the maximum of σ_{opd} does require the parameterized disturbance analysis.

The shortcoming of this harmonic model is that it does not lend itself to loop-shaping control design. Since the wheel speeds vary, any optical control system seeking to reject the wheel disturbances must do so over a band of wheel speeds, derived from the expected variations discussed above. Furthermore, since there is significant deviation from the model for different wheels (Figure 1 through Figure 16, [1]), the control system must have robust performance with respect to this variability among wheels. This necessity for broadband control design gave birth to a stochastic broadband disturbance model.

Stochastic Broadband RWA Disturbance Model

Several attempts have been made at deriving a broadband disturbance model from the discrete-frequency model. The first attempt was used in the "maximum jitter vs. frequency" envelope conceived by Laskin, San Martin, *et.al.* [3]. This model specified the maximum jitter resulting from the RWA disturbances at each frequency, over the ensemble of all possible wheel speeds. Although this envelope is not a disturbance model, it implies a broadband disturbance. Much later Rob Calvet generated an explicit over bounding broadband disturbance model. There are three problems with these models: 1) Since the actual RWA disturbances exist at only discrete-frequencies, the estimates with the over bounding broadband models are conservative, 2) the mathematical relationship to the actual discrete-frequency disturbances are somewhat convoluted, and 3) the broadband models are not easily generated for a restricted wheel speed range.

Gary Blackwood derived a modification to the Calvet model that equated the energy of the broadband excitation of a particular mode to the energy of a harmonic excitation of that same mode. This modification is still conservative, since the broadband model will excite all modes as if each was being excited by a harmonic, while the discrete-frequency disturbance will excite only several modes at any single wheel speed. Furthermore, this equi-energy model depends upon the half-power bandwidth and the residue of the structural modes in the given input-output relationship. In other words, the disturbance model depends on the structural properties.

In order to provide a mathematical interpretation of a broadband model that was independent of the structural properties, a stochastic broadband model was created. This model assumes that the wheel speed is a uniform random variable over some interval $[f_1, f_2]$, resulting in the broadband stochastic power spectral density, $\Phi_m(\omega)$, shown in Figure 2.

Since $m(t)$ is now a random process, $\Phi_m(\omega)$ is derived by taking the Fourier transform of the autocorrelation, $R_m(\tau)$ [4]:

$$\Phi_m(\omega) = \int_{-\infty}^{\infty} R_m(\tau) e^{-j\omega\tau} d\tau \tag{3}$$

where the autocorrelation is:

$$\begin{aligned} R_m(\tau) &= E\{m(t)m(t-\tau)\} \\ &= \sum_{i=1}^{n} \frac{C_i^2}{2} E\left\{f_{rwa}^4 \cos\left(2\pi h_i f_{rwa} \tau\right)\right\} \end{aligned} \tag{4}$$

The corresponding power spectral density is given in terms of the probability density function of the wheel speed, $f_p(u)$.

$$\Phi_m(\omega) = \sum_{i=1}^{n} \frac{\pi C_i^2 \omega^4}{2(2\pi h_i)^5} \left[f_p\left(\frac{\omega}{2\pi h_i}\right) + f_p\left(\frac{-\omega}{2\pi h_i}\right)\right] \tag{5}$$

The derivations of $R_m(\tau)$ and $\Phi_m(\omega)$ are given in the appendix.

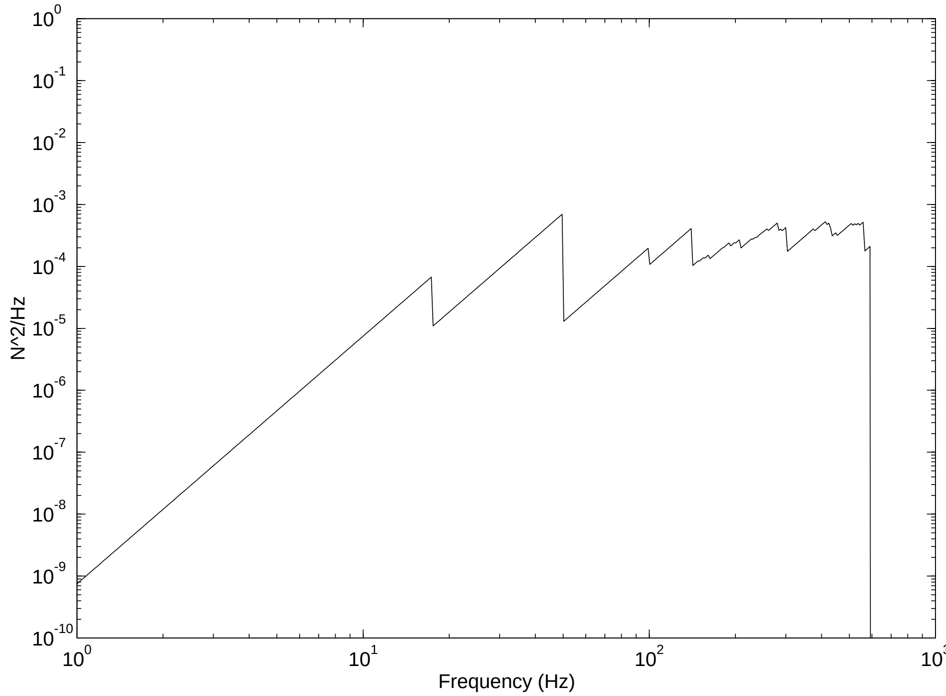

Figure 2: Stochastic broadband RWA radial force disturbance power spectral density assuming a uniform random variable wheel speed over the interval [0, 3000] RPM.

In this case, the probability density function for the reaction wheel speed is uniform:

$$f_p(u) = \begin{cases} \frac{1}{f_2 - f_1} & \text{for } f_1 < u \leq f_2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The resultant power spectral density is:

$$\begin{aligned} \Phi_m(\omega) &= \sum_{i=1}^{n} \Phi_{m_i}(\omega) \\ \Phi_{m_i}(\omega) &= \begin{cases} \frac{\pi C_i^2}{2(f_2 - f_1)(2\pi h_i)^5}\omega^4 & \text{for } 2\pi h_i f_1 < |\omega| < 2\pi h_i f_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7}$$

This is the description of the broadband power spectral density shown (for $f_1 = 0$Hz and $f_2 = 50$Hz) in Figure 2. Note that each component, $\Phi_{m_i}(\omega)$, is a straight line with slope 4 on a logarithmically scaled plot. Hence, Figure 2 is merely the superposition of these triangular component power spectral densities.

The power spectral density could be found from Eq. 5 for any assumed probability density f_p. As an example, the power spectral density for a Gaussian wheel speed is shown in Figure 3.

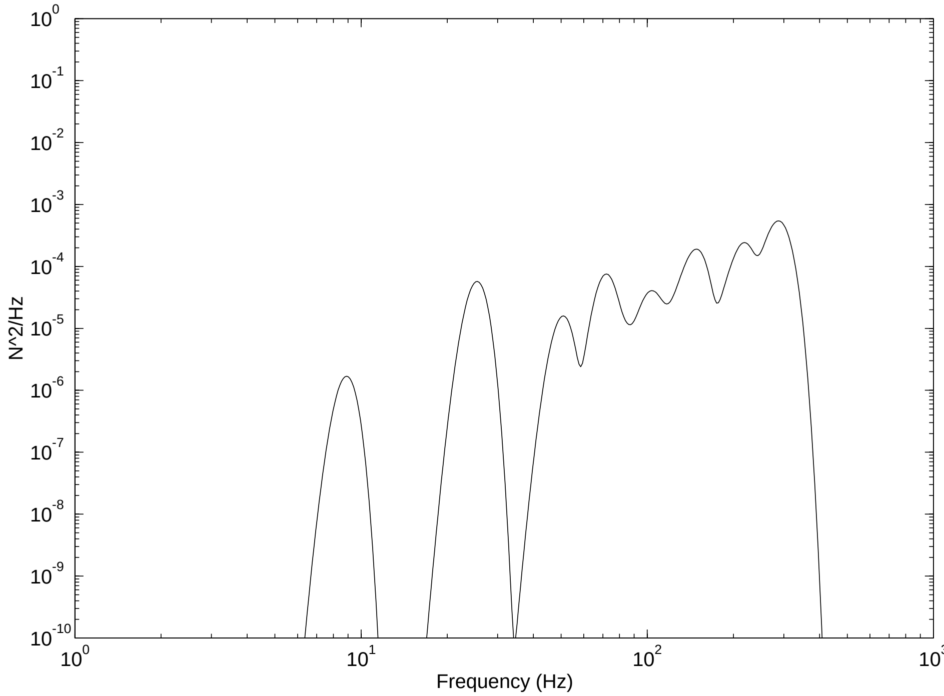

Figure 3: Stochastic broadband RWA radial force disturbance power spectral density assuming a Gaussian wheel speed with mean of 1500 RPM and standard deviation of 100 RPM.

As a result of this derivation, a broadband power spectral density can be generated by assuming that the reaction wheel speed is a random variable. Furthermore, that wheel speed can have a restricted range, by proper choice of the probability density function. In particular, a uniform random variable over the expected wheel speed range $[f_1,\ f_2]$ is proposed, leading to the power spectral density of Eq. 7.

Equivalence of $\mathcal{L}_2$ norm and Stochastic Standard Deviation

As mentioned above, the $\mathcal{L}_2$ norm can be calculated without going through the intermediate step of performing the parameterized disturbance analysis. This is possible since the $\mathcal{L}_2$ norm is equivalent to the standard deviation of the output resulting from the stochastic disturbance.

If $G(s)$ is the transfer function from the RWA disturbance to the output metric (call it OPD for consistency with Eq. 2), and if Φ_{nb} is the power spectral density of the *narrow band* RWA disturbance, then the root-mean-squared performance metric is given by Eq. 8 [4]. Note that the narrow band power spectral density is dependent upon the reaction wheel speed.

$$\sigma_{opd}^2(f_{rwa}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \Phi_{nb}(\omega, f_{rwa}) d\omega \tag{8}$$

Using this expression for the the performance metric and substituting into Eq. 2, the $\mathcal{L}_2$ norm becomes:

$$\|\sigma_{opd}\|_{\mathcal{L}_2}^2 = \frac{1}{2\pi f_{max}}\int_0^{f_{max}}\int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \Phi_{nb}(\omega, f_{rwa}) d\omega df_{rwa} \tag{9}$$

Now let $f_1 = 0$ and $f_2 = f_{max}$ in Eq. 6, then the $\mathcal{L}_2$ norm can be written as:

$$\|\sigma_{opd}\|_{\mathcal{L}_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_p(u) \int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \Phi_{nb}(\omega, u) d\omega du \tag{10}$$

Following the derivation in the Appendix, but with f_{rwa} deterministic, Φ_{nb} is found to be:

$$\Phi_{nb}(\omega, f_{rwa}) = \sum_{i=1}^{n}\left\{\frac{\pi C_i^2 f_{rwa}^4}{2}\left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right]\right\} \tag{11}$$

where $\delta(\omega - \omega_o)$ is the Dirac delta function and $\omega_o = 2\pi h_i f_{rwa}$ [6].

Next substitute Eq. 11 into Eq. 10, with the change of variables $f_{rwa} = u$. By reordering the integrations and summation, a final expression for the $\mathcal{L}_2$ norm is obtained.

$$\|\sigma_{opd}\|_{\mathcal{L}_2}^2 = \frac{1}{4}\sum_{i=1}^{n}\int_{-\infty}^{\infty} C_i^2 f_p(u) u^4 \int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \left[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right] d\omega du \tag{12}$$

Leaving aside the $\mathcal{L}_2$ norm for a moment, let's address the variance of the performance metric resultant from the broadband stochastic disturbance. The variance can be derived from the power spectral density according to Eq. 8, using the broadband power spectral density.

$$\tilde{\sigma}_{opd}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \Phi_m(\omega) d\omega \tag{13}$$

Here the standard deviation is written as $\tilde{\sigma}_{opd}$ in order to distinguish between the broadband and narrowband analyses.

Finally, substituting for $\Phi_m(\omega)$ according to Eq. 24 (see the appendix) and rearranging gives the result:

$$\tilde{\sigma}_{opd}^2 = \frac{1}{4}\sum_{i=1}^{n}\int_{-\infty}^{\infty} C_i^2 f_p(u) u^4 \int_{-\infty}^{\infty} |G(\jmath\omega)|^2 \left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right] d\omega du \tag{14}$$

Which is the same expression as Eq. 12. Thus, the $\mathcal{L}_2$ norm for the narrowband disturbance analysis is mathematically equivalent to the standard deviation for the stochastic broadband disturbance analysis, when the wheel speed is assumed to be a uniform random variable over the interval $[0,\ f_{max}]$.

Matlab Implementation

Both the discrete and broadband models have been implemented in Matlab and are available to users on the jpl-gnc net by including the `/opt/matlab-local/local4/models` toolbox in the matlabpath. The functions can be distributed to those who don't have access to the jpl-gnc net.

Discrete disturbance models are available for radial forces, axial force, and radial torques. The function names are `rwadist_rad.m`, `rwadist_axi.m`, and `rwadist_tor.m`, respectively. These functions generate either time domain or frequency domain information. For more specific descriptions, use the Matlab on-line help feature (*i.e.*, type `help rwadist_rad` in Matlab).

Similarly, broadband modeling functions are named `rwabroad_rad.m`, `rwabroad_axi.m`, and `rwabroad_tor.m`. These functions generate a power spectral density of the broadband disturbance for either a uniform random variable wheel speed or a Gaussian wheel speed. Again, details are given by the on-line help feature.

For questions or comments, please call me at 4-0615.

Appendix

The autocorrelation of the stochastic broadband RWA disturbance can be found by substituting for $m(t)$ according to Eq. 1 in the definition of autocorrelation.

$$\begin{aligned} R_m(\tau) &= E\left\{m(t)m(t-\tau)\right\} \\ &= E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} C_i C_j f_{rwa}^4 \sin(2\pi h_i f_{rwa} t + \phi_i)\sin(2\pi h_j f_{rwa}(t-\tau)+\phi_j)\right\} \end{aligned} \quad (15)$$

Trigonometric manipulation yields:

$$\begin{aligned} R_m(\tau) = E\Bigg\{\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{C_i C_j f_{rwa}^4}{2} &\left[\cos\left(2\pi f_{rwa}\left(h_i t - h_j\left(t-\tau\right)\right) + \phi_i - \phi_j\right)\right. \\ &\left.- \cos\left(2\pi f_{rwa}\left(h_i t + h_j\left(t-\tau\right)\right) + \phi_i + \phi_j\right)\right]\Bigg\} \end{aligned} \quad (16)$$

The expression for $R_m(\tau)$ is simplified by assuming that ϕ_i and ϕ_j are stochastically independent when $i \neq j$. This assumption is useful since:

$$E\left\{cos(\omega_o \tau + \phi_1 \pm \phi_2)\right\} = 0 \quad (17)$$

when ϕ_1 and ϕ_2 are independent and uniform over the interval $[0,\ 2\pi]$. Using the assumption of independence and the related observation, Eq. 16 is reduced to a single summation, since the expected value is zero when $i \neq j$.

$$R_m(\tau) = \sum_{i=1}^{n} \frac{C_i^2}{2} E\left\{f_{rwa}^4 \cos\left(2\pi h_i f_{rwa} \tau\right)\right\} \quad (18)$$

The autocorrelation can be solved by letting $z\left(f_{rwa}\right) = f_{rwa}^4 \cos\left(2\pi h_i \tau f_{rwa}\right)$ and noticing that $z\left(f_{rwa}\right)$ is a function of a single random variable. Rewriting $R_m(\tau)$ in terms of $z\left(f_{rwa}\right)$ illustrates the usefulness of this observation:

$$R_m(\tau) = \sum_{i=1}^{n} \frac{C_i^2}{2} E\left\{z\left(f_{rwa}\right)\right\} \quad (19)$$

The well known evaluation of $E\left\{z\left(f_{rwa}\right)\right\}$ yields [4]:

$$R_m(\tau) = \sum_{i=1}^{n} \frac{C_i^2}{2} \int_{-\infty}^{\infty} f_p(u) z(u) du \quad (20)$$

where $f_p(u)$ is the probability density function of the wheel speed.

The power spectral density is solved by substituting Eq. 20 into Eq. 3.

$$\Phi_m(\omega) = \sum_{i=1}^{n} \frac{C_i^2}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_p(u) u^4 \cos\left(2\pi h_i \tau u\right) e^{-j\omega\tau} du d\tau \quad (21)$$

Rearranging the order of integration results in an expression that is more easily evaluated.

$$\Phi_m(\omega) = \sum_{i=1}^{n} \frac{C_i^2}{2} \int_{-\infty}^{\infty} f_p(u) u^4 \int_{-\infty}^{\infty} \cos\left(2\pi h_i \tau u\right) e^{-j\omega\tau} d\tau du \tag{22}$$

The inner integral of Eq. 22 is merely the Fourier transform of the cosine function, $\mathcal{F}\{\cos(\omega_o \tau)\}$. Substituting for the Fourier transform [5]:

$$\mathcal{F}\{\cos(\omega_o \tau)\} = \pi \left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right] \tag{23}$$

leads to:

$$\Phi_m(\omega) = \sum_{i=1}^{n} \frac{\pi C_i^2}{2} \int_{-\infty}^{\infty} f_p(u) u^4 \left[\delta\left(\omega - 2\pi h_i u\right) + \delta\left(\omega + 2\pi h_i u\right)\right] du \tag{24}$$

where $\delta(\omega)$ designates the Dirac delta function [6].

Using the sifting property of the Dirac delta function, the power spectral density is expressed in closed form, as a function of the probability density function of the reaction wheel speed, $f_p(u)$.

$$\Phi_m(\omega) = \sum_{i=1}^{n} \frac{\pi C_i^2 \omega^4}{2(2\pi h_i)^5} \left[f_p\left(\frac{\omega}{2\pi h_i}\right) + f_p\left(\frac{-\omega}{2\pi h_i}\right)\right] \tag{25}$$

References

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